STANDARD MULTIGRADED HIBI RINGS AND CARTWRIGHT–STURMFELS IDEALS

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This talk is based on joint work with Koji Matsushita [4].

We assume that all posets and lattices appearing in this talk are finite. Let P be a poset equipped with a partial order $\leq_P$. We assume that P is compatible with the usual ordering of $[1,n]$, that is, if $i <_P j$, then $i < j$ in $\mathbb{Z}$. A *poset ideal* of P is a subset α of P satisfying that, whenever $a \in \alpha$ and $b \in P$ with $b \leq_P a$, one has $b \in \alpha$. Let $L(P)$ be the set of poset ideals of P. Then, the set $L(P)$ forms a distributive lattice ordered by inclusion. Conversely, the poset P can be recovered from $L(P)$. This is well known as Birkhoff's representation theorem [1]. Let $S_{L(P)} = \Bbbk[x_\alpha : \alpha \in L(P)]$ be the polynomial ring over $\Bbbk$ and $I_{L(P)}$ be the ideal of $S_{L(P)}$ generated by

$$\mathcal{F}_{L(P)} \coloneqq \{f_{\alpha,\beta} : \alpha, \beta \in L(P) \text{ are incomparable.}\},$$

where $f_{\alpha,\beta} \coloneqq x_\alpha x_\beta - x_{\alpha\cap\beta}x_{\alpha\cup\beta}$ for incomparable elements $\alpha, \beta \in L(P)$. The ring $\Bbbk[P] := S_{L(P)}/I_{L(P)}$ is called *Hibi ring* of P, introduced in [3].

We introduce a standard multigrading on the Hibi ring and study its properties. First, for a nonnegative integer m, we define a standard $\mathbb{Z}^{m+1}$-multigrading on $S_{L(P)}$, that is, each variable of $S_{L(P)}$ has degree $\mathbf{e}_i$ for some $i \in [0,m]$ where $\mathbf{e}_i$ stands for the ith unit vector of $\mathbb{Z}^{m+1}$. Let $C = \{c_1 <_P \cdots <_P c_\ell\}$ be a chain of P and $C_i := \{c_1, \dots, c_i\}$ for $i = 0, 1, \dots, \ell$ where we let $C_0 := \emptyset$. Moreover, let $\mathfrak{p}(C) := \{C_0, C_1, \dots, C_\ell\}$ and let $f_C : \mathfrak{p}(C) \to [0,m]$ be a map. Note that for any poset ideal α of P, the set $\alpha \cap C$ belongs to $\mathfrak{p}(C)$. We define the standard $\mathbb{Z}^{m+1}$-multigrading on $S_{L(P)}$ by setting $\deg_{f_C}(x_\alpha) := \mathbf{e}_{f_C(\alpha\cap C)}$.

Theorem 1. *Work with the same notation as above. Then $I_{L(P)}$ is homogeneous. Conversely, suppose that $S_{L(P)}$ is standard $\mathbb{Z}^{m+1}$-multigraded with a degree map* $\deg$ *such that $I_{L(P)}$ is homogeneous. Then there exist a chain C of P and a map f_C : $\mathfrak{p}(C) \to [0,m]$ with* $\deg = \deg_{f_C}$.

We compute multigraded Hilbert series and multidegree polynomials of multigraded Hibi rings in terms of posets and distributive lattices. Furthermore, we characterize Hibi ideals that are Cartwright–Sturmfels ideals.

For a standard $\mathbb{Z}^{m+1}$-graded polynomial ring S and a homogeneous ideal I, the residue ring $S/I = \bigoplus_{\mathbf{a}\in\mathbb{Z}^{m+1}_{\geq 0}} S_\mathbf{a}/I_\mathbf{a}$ can be considered as a $\mathbb{Z}^{m+1}$-graded ring. The *multigraded Hilbert series* of S/I is

$$\mathrm{HS}(S/I;\mathbf{t}) = \sum_{\mathbf{a}\in\mathbb{Z}^{m+1}_{\geq 0}} \dim_\Bbbk (S/I)_\mathbf{a} \mathbf{t}^\mathbf{a} = \frac{\mathcal{K}(S/I;\mathbf{t})}{\prod_{i=1}^m (1-\mathbf{t}^{\deg(x_i)})}.$$

The numerator $\mathcal{K}(S/I;\mathbf{t}) \in \mathbb{Z}[\mathbf{t}]$ is called the *$\mathcal{K}$-polynomial* of S/I. The sum $\mathcal{C}(S/I;\mathbf{t}) \in \mathbb{Z}[\mathbf{t}]$ of all terms in $\mathcal{K}(S/I;1-\mathbf{t})$ having total degree $\operatorname{codim}(S/I)$ is called *multidegree polynomial* of S/I.

Let n and $m_1,\dots,m_n$ be positive integers. Let $S = \Bbbk[x_{i,j} : 1 \le j \le n, 1 \le i \le m_j]$ be a standard $\mathbb{Z}^n$-graded polynomial ring with $\deg(x_{i,j}) = \mathbf{e}_j$. Let $G = \mathrm{GL}_{m_1}(\Bbbk) \times \cdots \times \mathrm{GL}_{m_n}(\Bbbk)$. We assume that G acts on S. Precisely, let $g = (g^{(1)},\dots,g^{(n)}) \in G$ acts on variables in S by

$$g \cdot x_{i,j} = \sum_{k=1}^{m_j} g_{k,i}^{(j)} x_{k,j}.$$

Let $B = \mathrm{B}_{m_1}(\Bbbk) \times \cdots \times \mathrm{B}_{m_n}(\Bbbk)$ be the *Borel subgroup* of G, where each $\mathrm{B}_{m_j}(\Bbbk)$ consists of upper triangular matrices in $\mathrm{GL}_{m_j}(\Bbbk)$. We say that a homogeneous ideal $J \subset S$ is *Borel-fixed* if $g \cdot J = J$ for all $g \in B$.

Definition 2 ([2, Definition 2.4])**.** A homogeneous ideal I is *Cartwright–Sturmfels* if there exists a squarefree multigraded Borel-fixed monomial ideal J such that $\mathrm{HS}(S/I;\mathbf{t}) = \mathrm{HS}(S/J;\mathbf{t})$.

Theorem 3. *Let P be a poset and $L(P)$ be the distributive lattice associated with P. Let $C = \{c_1 <_P \dots <_P c_\ell\}$ be a chain of P. We assume that $S_{L(P)}$ is multigraded by C. Then, $I_{L(P)}$ is a Cartwright–Sturmfels ideal if and only if $P \setminus C$ is a chain of P.*

References

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